Geopolitical Disaster Risk, Oil Supply Shocks, and the Carry-Crash Duality in Exchange Rates

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Abstract

Why does the correlation between oil prices and oil-exporter exchange rates weaken or reverse during geopolitical crises? A continuous-time two-country model with CES preferences, Epstein-Zin recursive utility, oil supply jump-diffusion, and asymmetric disaster exposure delivers three results. First, the sign of the demand-driven oil-currency covariance depends on the elasticity of substitution between oil and the numeraire. When oil and the numeraire are complements ($\sigma < \sigma^*$), demand booms raise the oil price and appreciate the exporter's currency through expenditure-share amplification, producing a positive between-disaster oil-currency correlation. The Cobb-Douglas special case ($\sigma = 1$) is the knife-edge where the demand channel vanishes. Second, geopolitical disasters add a negative contribution proportional to disaster intensity, so the total correlation shifts from positive in calm times to negative under geopolitical stress. Third, Epstein-Zin preference separation generates a carry-crash duality with a closed-form commodity disaster beta that disciplines the Farhi-Gabaix resilience parameter using oil dependence and disaster severity.

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1 Introduction

Oil-exporter currencies and oil prices move together. Over most sample periods, the Canadian dollar, the Norwegian krone, and the Russian ruble appreciate when oil prices rise and depreciate when oil prices fall (Cashin et al., 2004; Chen et al., 2010). Yet this positive correlation weakens and sometimes reverses during geopolitical crises. Bjornland et al. (2020) document that the oil-exchange rate relationship becomes unstable during periods of elevated geopolitical tension, and Dodd et al. (2024) find outright sign reversal under high geopolitical risk. What mechanism produces a positive oil-currency correlation in calm times that turns negative under geopolitical stress?

A continuous-time two-country model with CES preferences, Epstein-Zin recursive utility, oil supply jump-diffusion, and asymmetric disaster exposure generates a state-dependent oil-currency correlation whose sign depends on the elasticity of substitution between oil and the numeraire and on the level of geopolitical risk. First, the sign of the demand-driven oil-currency covariance depends on whether oil and the numeraire are complements or substitutes. When the elasticity of substitution σ falls below a threshold σ^*, demand booms raise the oil price and redistribute toward the exporter through an expenditure-share amplification mechanism, generating a positive between-disaster oil-currency correlation. The Cobb-Douglas special case ($\sigma = 1$) is the knife-edge where the demand channel vanishes and only the negative supply channel remains. Second, geopolitical disasters are supply events that spike the oil price and crash the exporter's currency. The disaster contribution to the oil-currency covariance is negative and proportional to disaster intensity, which rises with geopolitical risk. The total correlation therefore shifts from positive in calm times to negative under geopolitical stress. Third, Epstein-Zin preference separation generates a carry-crash duality: the exporter's currency appreciates between disasters (the carry) and crashes at disaster (the crash), with the carry premium governed by a closed-form commodity disaster beta (CDB).

The model builds on Farhi and Gabaix (2016), who show that country-specific disaster exposure determines currency risk premia and explains the forward premium puzzle. Oil supply diffusion creates between-disaster oil-exchange rate covariance absent in Farhi and Gabaix (2016); oil dependence of the exporter's non-oil sector replaces the free resilience parameter in Farhi and Gabaix (2016) with a calibratable function; and CES preferences reveal how the sign of the demand-driven oil-currency correlation depends on the elasticity of substitution. The model connects to Kilian et al. (2024), who study geopolitical oil price risk in a DSGE framework without exchange rates, and to Ready et al. (2017), who generate a positive commodity-currency correlation through capital reallocation in a production economy. The CES demand channel operates through a different mechanism (expenditure-share amplification under complementary goods) and delivers an explicit threshold condition on preferences. The carry-crash duality connects to the broader carry trade literature of Lustig et al. (2011), providing a structural decomposition of the carry factor loading in terms of oil dependence and disaster severity.

The CDB aggregates the disaster resilience differential, disaster severity, and oil dependence into a single quantity that prices the exporter's currency risk. The CDB replaces the free resilience parameter in Farhi and Gabaix (2016) with a function of calibratable parameters, connecting the currency risk premium literature to the structural oil economics literature of Kilian (2009). An incomplete markets extension enriches the disaster structure with a non-oil shock orthogonal to the oil price and restricts the exporter to trading bonds and oil futures. Oil futures span both diffusion and jump risk from the oil price, but the residual spanning wedge is quantitatively negligible: incomplete spanning cannot resolve the dual-γ calibration tension. Sanctions that remove futures access create a structurally distinct welfare cost by eliminating an insurance instrument rather than shifting a parameter, and a sharp welfare-premium disconnect emerges: the exporter faces welfare costs exceeding 11% of permanent consumption while the carry premium changes by less than 1 basis point per year.

Section 2 presents the model. Section 3 derives the main results under CES preferences, with the Cobb-Douglas model as a special case. Section 4 extends the model to incomplete markets with non-oil disaster shocks and restricted asset span. Section 5 presents empirical evidence (Figures 1 and 2), testable predictions, welfare implications, and limitations. Section 6 concludes. The appendix contains supplementary derivations.

2 Model

2.1 Environment

Two countries, E (oil exporter) and I (oil importer), operate in continuous time over $[0, \infty)$. Two goods exist: oil and a numeraire. Each country has a representative agent.

Endowments. Country E receives oil Y_t^O and numeraire $Y_t^{N,E}$. Country I receives numeraire $Y_t^{N,I}$ only.

Oil supply follows a jump-diffusion:

$$\frac{dY_t^O}{Y_{t^-}^O} = g_O\, dt + \sigma_S\, dB_t^S - Z\, dJ_t \tag{1}$$

where $g_O \geq 0$ is trend growth, $\sigma_S > 0$ is diffusion volatility, B_t^S is a standard Brownian motion, $Z \sim F$ on $(0, \bar{Z})$ with $\bar{Z} < 1$ is random disaster severity, and J_t is a Poisson process with stochastic intensity $\lambda(G_t)$ defined below.

Numeraire endowments are

$$Y_t^{N,I} = \bar{Y}^{N,I} \cdot D_t \cdot e^{g_N t}, \qquad Y_t^{N,E} = \bar{Y}^{N,E} \cdot \left(\frac{Y_t^O}{\bar{Y}^O}\right)^\eta \cdot e^{g_N t} \tag{2}$$

where g_N is deterministic trend growth, $\eta \in [0,1]$ measures the oil dependence of the exporter's non-oil sector, and D_t is a demand shock:

$$\frac{dD_t}{D_t} = \sigma_D \, dB_t^D \tag{3}$$

with B_t^D independent of B_t^S and W_t, and $\sigma_D \geq 0$. The Cobb-Douglas special case sets $\sigma_D = 0$.

Geopolitical risk. A state variable $G_t \geq 0$ captures geopolitical tension:

$$dG_t = \kappa(\bar{G} - G_t)\, dt + \sigma_G \, dW_t \tag{4}$$

where $\kappa > 0$ governs mean reversion, $\bar{G} > 0$ is the long-run mean, $\sigma_G > 0$ is diffusion volatility, and W_t is a standard Brownian motion. The process is reflected at zero. For the baseline calibration, G_t falls below zero less than 3.5% of the time under the unreflected Ornstein-Uhlenbeck, so reflection has negligible impact on the stationary distribution.

Disaster process. Geopolitical disasters arrive with intensity

$$\lambda(G_t) = \lambda_0 + \lambda_1 G_t \tag{5}$$

where $\lambda_0 > 0$ and $\lambda_1 > 0$. The OU dynamics with mean reversion $\kappa > 0$ keep G_t bounded in probability: the stationary distribution is $N(\bar{G}, \sigma_G^2/(2\kappa))$, so $\lambda(G_t)$ has finite moments of all orders under the stationary measure. In particular, $\mathbb{E}[\lambda(G_t)] = \lambda_0 + \lambda_1 \bar{G}$ is finite and the expected number of disasters over any finite horizon is bounded. For any $G_{\max}$ such that $\Pr(G_t > G_{\max}) < \epsilon$, the intensity $\lambda(G_{\max})$ provides a practical upper bound. At the baseline calibration, $G_t > 3\bar{G}$ with probability less than 0.1%, giving $\lambda_{\max} \approx 0.025$. This ensures that the integrability conditions required by Duffie and Skiadas (1994) for existence of equilibrium with bounded-intensity jump processes are satisfied; see Appendix A.1 for details.

Independence. The Brownian motions B_t^S, B_t^D, and W_t are mutually independent. The Poisson process J_t is independent of all three.

Preferences. Both countries have CES preferences over oil and the numeraire:

$$C^i = \left[\alpha^{1/\sigma}(O^i)^{(\sigma-1)/\sigma} + (1-\alpha)^{1/\sigma}(N^i)^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)} \tag{6}$$

where $\sigma > 0$ is the elasticity of substitution between oil and the numeraire, $\alpha \in (0,1)$ is the oil share parameter, O^i is oil consumption, and N^i is numeraire consumption by country i. The Cobb-Douglas specification ($\sigma = 1$) is a special case in which $C^i = (O^i)^\alpha (N^i)^{1-\alpha}$.

2.2 Agents' problem

Both countries have Epstein-Zin recursive utility in the Duffie and Epstein (1992) continuous-time formulation:

$$V_t^i = \mathbb{E}_t\left[\int_t^\infty f(C_s^i, V_s^i)\, ds\right] \tag{7}$$

with normalized aggregator

$$f(C,V) = \frac{\beta}{1-1/\psi}\left[\frac{C^{1-1/\psi}}{((1-\gamma)V)^{1/\theta-1}} - (1-\gamma)V\right] \tag{8}$$

where $\gamma > 0$ is relative risk aversion, $\psi > 0$ is the intertemporal elasticity of substitution (IES), $\beta > 0$ is the rate of time preference, and $\theta \equiv (1-\gamma)/(1-1/\psi)$.

Each agent maximizes V_0^i subject to the Arrow-Debreu budget constraint:

$$\mathbb{E}\left[\int_0^\infty \xi_t C_t^i\, dt\right] \le \mathbb{E}\left[\int_0^\infty \xi_t I_t^i\, dt\right] \tag{9}$$

where ξ_t is the state-price density and I_t^i is country i's income.

2.3 Exchange rate

The real exchange rate is

$$e_t \equiv \frac{\xi_t^I}{\xi_t^E} \tag{10}$$

An increase in e_t means country E's currency appreciates.

2.4 Equilibrium

Definition 1 (Competitive Equilibrium)**.** *A competitive equilibrium consists of state-price density processes* $\{\xi_t^i\}$*, consumption allocations* $\{C_t^i\}$*, an oil price process* $\{p_t\}$*, and an exchange rate process* $\{e_t\}$ *such that:*

1. *Each country's consumption plan maximizes Epstein-Zin utility subject to the Arrow-Debreu budget constraint.*
2. *The oil market clears:* $O_t^E + O_t^I = Y_t^O$.
3. *The numeraire market clears:* $N_t^E + N_t^I = Y_t^{N,E} + Y_t^{N,I}$.
4. *Asset markets clear: the risk-free bond is in zero net supply.*
5. *The exchange rate satisfies no-arbitrage:* $e_t = \xi_t^I / \xi_t^E$.

Existence follows from Duffie and Skiadas (1994) and Schroder and Skiadas (1999) for continuous-time Epstein-Zin economies with jump-diffusion processes. The key integrability condition requires that endowment processes have finite moments under the physical measure; Appendix A.1 verifies this condition for the jump-diffusion oil supply process with stochastic but bounded-in-probability intensity $\lambda(G_t)$. Uniqueness follows from the concavity of the aggregator: with complete markets, two representative agents, and homothetic preferences, the equilibrium is indexed by a single Pareto weight and the First Welfare Theorem applies.

2.5 Oil price and consumption allocations

Oil price under CES. The oil expenditure share is endogenous under CES:

$$s_O(p) = \frac{\alpha p^{1-\sigma}}{\alpha p^{1-\sigma} + (1-\alpha)} \tag{11}$$

Oil market clearing under the small exporter approximation ($Y_t^{N,E} \ll Y_t^{N,I}$) gives $s_O(p)\, Y_t^{N,I}/p \approx Y_t^O$. Let $x_t \equiv \ln(Y_t^{N,I}/Y_t^O)$ denote the log demand-supply ratio. The oil price elasticity with respect to x_t is

$$\varepsilon_p \equiv \frac{d \ln p}{dx} = \frac{1}{\sigma + (1-\sigma)s_O} \tag{12}$$

When $\sigma = 1$: $\varepsilon_p = 1$. When $\sigma < 1$: $\varepsilon_p > 1$. When $\sigma > 1$: $\varepsilon_p < 1$.

Under Cobb-Douglas ($\sigma = 1$, $\varepsilon_p = 1$, $s_O = \alpha$), each country spends fraction α of total income on oil. Oil market clearing yields

$$p_t = \frac{\alpha}{1-\alpha}\frac{Y_t^N}{Y_t^O} \tag{13}$$

where $Y_t^N \equiv Y_t^{N,E} + Y_t^{N,I}$ is aggregate numeraire.

Small exporter approximation. The exporter's numeraire endowment is small relative to the importer's: $Y_t^{N,E} \ll Y_t^{N,I}$, so $Y_t^N \approx Y_t^{N,I}$. This approximation introduces errors of order ϕ_E (the exporter's share of world numeraire). Appendix A.3 provides a formal error analysis showing the approximation error in the oil-currency covariance is $O(\phi_E^2)$, not $O(\phi_E)$, because the first-order correction enters symmetrically and cancels in the covariance. At the baseline $\phi_E = 0.6$, the bound $O(\phi_E^2) = O(0.36)$ is nontrivial; the appendix verifies numerically that the sign of the between-disaster covariance is preserved under both the Cobb-Douglas and CES specifications.

Between disasters, the oil price dynamics are

$$d\ln p_t = \mu_p\,dt + \varepsilon_p\sigma_D\,dB_t^D - \varepsilon_p\sigma_S\,dB_t^S \tag{14}$$

Under Cobb-Douglas ($\varepsilon_p = 1$, $\sigma_D = 0$): $d\ln p_t = (g_N - g_O + \frac{1}{2}\sigma_S^2)\,dt - \sigma_S\,dB_t^S$. At disaster realization, $\Delta\ln p_t = \varepsilon_p^{\text{jump}}\ln(1/(1-Z)) > 0$: oil supply drops and the price spikes.

Notation for disaster exposures. Define effective disaster exposures $\ell_I \equiv \alpha$ and $\ell_E \equiv \alpha + \phi_E\eta$, where ϕ_E is the exporter's non-oil income share. The notation ℓ (for "loading") replaces the d notation used in earlier drafts, to avoid confusion with the differential operator in expressions such as dB and dt.

Between disasters:

$$d\ln C_t^I = \mu_I\,dt + c_I^S\,\sigma_S\,dB_t^S + c_I^D\,\sigma_D\,dB_t^D \tag{15}$$

$$d\ln C_t^E = \mu_E\,dt + c_E^S\,\sigma_S\,dB_t^S + c_E^D\,\sigma_D\,dB_t^D \tag{16}$$

Under Cobb-Douglas ($\varepsilon_p = 1$): $c_I^S = \ell_I = \alpha$, $c_E^S = \ell_E = \alpha + \phi_E\eta$, $c_I^D = 1$, and $c_E^D = 1 - \phi_E$. Under general CES, the loadings depend on ε_p and s_O; Section 3.2 provides the expressions. At disaster: $\Delta\ln C_t^i = \ell_i\ln(1-Z)$ for $i \in \{E, I\}$. The exporter is more exposed to oil shocks ($\ell_E > \ell_I$ when $\eta > 0$), and this asymmetry drives all results. Appendix A.2 provides the full derivation.

3 Results

3.1 Exchange rate dynamics

Under the Duffie-Epstein formulation, the log stochastic discount factor of country i loads on consumption shocks with coefficient $-\gamma$ and on the wealth-consumption ratio with coefficient $1/\psi - \gamma$. The wealth-consumption ratio $\nu_t^i = W_t^i / C_t^i$ depends on G_t through the disaster intensity:

$$\ln \nu_t^i \approx \bar{\nu}^i + \frac{\Phi^i}{\delta\chi}\lambda(G_t) \tag{17}$$

where $\Phi^i \equiv \mathbb{E}[(1-Z)^{-\gamma\ell_i}] - 1$ is the disaster resilience of country i, $\delta \equiv \beta + (1/\psi - 1)\bar{g}$ is the effective discount rate, and $\chi \equiv 1 + \delta/\kappa$ is the mean-reversion adjustment. Since $\ell_E > \ell_I > 0$ and $\gamma > 0$, the resilience ordering $\Phi^E > \Phi^I > 0$ holds.

Combining the consumption channel and the wealth-consumption ratio channel, the exchange rate dynamics are

$$d\ln e_t = \gamma(c_E^S - c_I^S)\sigma_S\, dB_t^S + \gamma(c_E^D - c_I^D)\sigma_D\, dB_t^D + \beta_e \sigma_G\, dW_t + \mu_e\, dt \tag{18}$$

where the carry sensitivity is

$$\beta_e \equiv \frac{(\gamma - 1/\psi)(\Phi^E - \Phi^I)\lambda_1}{\delta\chi} \tag{19}$$

At disaster realization:

$$\Delta \ln e_t = \gamma(\ell_E - \ell_I)\ln(1-Z) = \gamma\phi_E\eta\ln(1-Z) < 0 \tag{20}$$

Three independent stochastic factors drive the exchange rate between disasters: oil supply shocks (B_t^S), demand shocks (B_t^D), and geopolitical risk shocks (W_t). At disaster, the exporter's currency crashes.

Remark 1. *The aggregator derivative f_V introduces a correction to β_e. The effective carry sensitivity is $\beta_e(1 + \eta_{fV})$ (Lemma 16, Appendix), where $\eta_{fV} > 0$ amplifies β_e by less than 1% at the baseline calibration ($\gamma = 5$, $\psi = 2$). Under CES with $\varepsilon_p > 1$, the amplified consumption loadings feed back into the wealth-consumption ratio. Appendix A.4 verifies that the f_V correction remains below 2% for $\sigma \geq 0.20$, because ε_p amplifies both the numerator and denominator of η_{fV} proportionally. All numerical results use the uncorrected β_e.*

3.2 CES model: consumption loadings and the oil price elasticity

Proposition 2 (Oil Price Elasticity under CES). *Let $x_t \equiv \ln(Y_t^{N,I}/Y_t^O)$ denote the log demand-supply ratio and s_O the oil expenditure share at steady state. The elasticity of the log oil price with respect to x_t is*

$$\varepsilon_p = \frac{1}{\sigma + (1-\sigma)s_O} \tag{21}$$

When $\sigma = 1$: $\varepsilon_p = 1$. When $\sigma < 1$: $\varepsilon_p > 1$. When $\sigma > 1$: $\varepsilon_p < 1$.

Proof. Oil market clearing under CES gives $s_O(p)\, Y^{N,I}/p = Y^O$ where $s_O(p) = \alpha p^{1-\sigma}/[\alpha p^{1-\sigma} + (1-\alpha)]$. Taking log-derivatives: $(d\ln s_O/d\ln p)(d\ln p/dx) + 1 - d\ln p/dx = 0$. Since $d\ln s_O/d\ln p = (1-\sigma)(1-s_O)$, solving for $d\ln p/dx$ gives (21). □

Under CES, the consumption loadings on the supply and demand shocks are:

$$\Delta_S \equiv c_E^S - c_I^S = \phi_E\eta - (1-\phi_E)(\varepsilon_p - 1) \tag{22}$$

$$\Delta_D \equiv c_E^D - c_I^D = (1-\phi_E)(1-\sigma)(1-s_O)\varepsilon_p - \phi_E \tag{23}$$

The importer's loadings are $c_I^S = s_O\varepsilon_p$ and $c_I^D = 1 - s_O\varepsilon_p$. The exporter's loadings follow from income-share-weighted aggregation (Appendix A.7).

Under Cobb-Douglas ($\varepsilon_p = 1$): $\Delta_S = \phi_E\eta > 0$ and $\Delta_D = -\phi_E < 0$. A supply expansion appreciates the exporter's currency (positive loading on B^S), while a demand boom has a negative demand-channel contribution to the oil-currency covariance.

Under CES with $\sigma < 1$: $\varepsilon_p > 1$ and two departures from the Cobb-Douglas case emerge. First, the supply-side differential Δ_S shrinks and can turn negative under strong complementarity ($\varepsilon_p > 1 + \phi_E\eta/(1-\phi_E)$), because a supply expansion reduces the exporter's oil revenue when demand is inelastic ($\varepsilon_p > 1$ means the price falls more than quantity rises). Second, the demand-side differential Δ_D turns positive when $(1-\sigma)(1-s_O)\varepsilon_p > \phi_E/(1-\phi_E)$: a demand boom raises the oil price and redistributes toward the exporter through expenditure-share amplification.

3.3 Oil-currency covariation

Proposition 3 (Oil-Currency Covariation). *Under CES preferences with $\sigma_S > 0$ and $\eta > 0$:*

(a) The between-disaster oil-currency covariance is

$$\mathrm{Cov}_t(d\ln p_t, d\ln e_t)\big|_{no\ disaster} = \gamma\varepsilon_p \left[-\Delta_S\sigma_S^2 + \Delta_D\sigma_D^2\right] dt \tag{24}$$

(b) *At disaster realization, the cross-moment is*

$$\mathbb{E}[\Delta \ln p_t \cdot \Delta \ln e_t \mid \textit{disaster}] = -\gamma\phi_E\eta \cdot \varepsilon_p^{jump} \cdot \mathbb{E}[(\ln(1-Z))^2] < 0 \tag{25}$$

where $\varepsilon_p^{jump} > 0$ *is the oil price elasticity at the disaster realization.*

(c) *Under Cobb-Douglas* ($\sigma = 1$, $\sigma_D = 0$)*: the between-disaster covariance reduces to* $-\gamma\phi_E\eta\sigma_S^2\,dt < 0$*, and the between-disaster correlation is* $\rho_{pe}^{between} = -\gamma\phi_E\eta\sigma_S/\sqrt{\gamma^2\phi_E^2\eta^2\sigma_S^2 + \beta_e^2\sigma_G^2} < 0$*, a negative constant independent of* G_t*.*

Proof. (a) Between disasters, $d\ln p$ and $d\ln e$ load on independent Brownian motions B^S and B^D, and W contributes no covariance with p. Computing the instantaneous covariance gives (24). (b) At disaster, only oil supply jumps. The exporter's non-oil income drops by $\eta\ln(1-Z)$, while the oil revenue windfall is bounded by s_O times the price increase. For s_O small (baseline: 0.07), the non-oil loss dominates. (c) Setting $\sigma = 1$ recovers $\varepsilon_p = 1$, $\Delta_S = \phi_E\eta$, and $\sigma_D = 0$, giving $-\gamma\phi_E\eta\sigma_S^2\,dt$. □

A positive supply shock lowers the oil price; under Cobb-Douglas the exporter benefits through its non-oil sector (η), while under CES with $\sigma < 1$ the amplified price response ($\varepsilon_p > 1$) reduces oil revenue, offsetting the non-oil benefit. A positive demand shock raises the oil price; under Cobb-Douglas the fixed expenditure share makes the demand channel negative, while under CES with complements the rising expenditure share redistributes toward the exporter.

3.4 Carry-crash duality

Proposition 4 (Carry-Crash Duality)**.** *With* $\gamma > 1$, $\psi > 1$, *and* $\ell_E > \ell_I$:

(a) *Between disasters, higher geopolitical risk appreciates the exporter's currency:*

$$\left.\frac{\partial \ln e_t}{\partial G_t}\right|_{\textit{between disasters}} = \beta_e > 0 \tag{26}$$

(b) *At disaster realization, the exporter's currency depreciates:*

$$\Delta \ln e_t\big|_{\textit{disaster}} = \gamma\phi_E\eta\ln(1-Z) < 0 \tag{27}$$

(c) *Under CRRA* ($\gamma = 1/\psi$)*:* $\beta_e = 0$*. The carry channel vanishes. EZ preference separation is necessary for carry but not for the supply channel or the disaster crash.*

Proof. (a) From (17), $\ln e_t$ loads on G_t with coefficient $\beta_e > 0$ when $\gamma > 1/\psi$ and $\Phi^E > \Phi^I$. (b) At disaster, $\Delta \ln C^i = \ell_i \ln(1 - Z)$ and the wealth-consumption ratio does not jump to first order, giving $\Delta \ln e = \gamma(\ell_E - \ell_I) \ln(1 - Z) < 0$. (c) Setting $\gamma = 1/\psi$ gives $\theta = 1$ and $\beta_e = 0$. □

The carry-crash duality arises from EZ preference separation. Between disasters, the exporter's riskier consumption claims trade at a larger discount, appreciating its currency. At disaster, the exporter's consumption falls more and its marginal utility spikes, crashing the currency. The duality holds regardless of σ: it depends only on $\ell_E - \ell_I$.

3.5 Sign of the demand channel

Proposition 5 (Sign of the Demand Channel)**.** *The oil-currency covariance from demand shocks satisfies* $\mathrm{Cov}(d \ln p, d \ln e)|_D = \gamma \varepsilon_p \sigma_D^2 \, \Delta_D \, dt$*, where* Δ_D *is defined in* (23)*.*

(a) When $\sigma = 1$ *(Cobb-Douglas):* $\Delta_D = -\phi_E < 0$.

(b) When $\sigma < 1$ *and the complementarity satisfies* $(1-\sigma)(1-s_O)\varepsilon_p > \phi_E/(1-\phi_E)$*:* $\Delta_D > 0$ *and the demand channel produces a positive oil-currency covariance.*

(c) The threshold elasticity σ^* *satisfies*

$$(1 - \phi_E)(1 - \sigma^*)(1 - s_O)\varepsilon_p(\sigma^*) = \phi_E \tag{28}$$

For $\sigma < \sigma^*$*: the demand-driven covariance is positive. For* $\sigma > \sigma^*$*: negative.*

Proof. The oil price loads on dB^D with coefficient $\varepsilon_p \sigma_D > 0$. The exchange rate loads on dB^D through $\gamma \Delta_D \sigma_D$. Parts (a)–(c) follow from the sign of $(1 - \sigma)$ and the monotonicity of ε_p in σ. □

When oil and the numeraire are complements ($\sigma < 1$), a demand-driven oil price increase raises the oil expenditure share. The exporter collects all oil revenue and benefits more than proportionally; the importer is squeezed. Under Cobb-Douglas ($\sigma = 1$), the expenditure share is constant and the demand channel vanishes. Table 1 reports the threshold σ^* for several values of ϕ_E.

Table 1: Threshold Elasticity σ^* by Exporter Income Share ϕ_E

ϕ_E	0.3	0.4	0.5	0.6
σ^*	0.590	0.524	0.441	0.355

Countries with smaller ϕ_E (more diversified non-oil sectors, or smaller non-oil economies relative to the world) face a higher σ^*, meaning the positive demand channel activates for a wider range of elasticities. At $\phi_E = 0.6$ (baseline), the threshold is $\sigma^* = 0.355$. Empirical estimates of the oil-numeraire elasticity of substitution range from 0.2 to 0.5 (Backus and Crucini, 2000; Bodenstein et al., 2011). Values at the lower end ($\sigma \approx 0.2$–0.3) satisfy $\sigma < \sigma^*$ and activate the positive demand channel.

3.6 Sign reversal with geopolitical risk

Proposition 6 (Sign Reversal under CES)**.** *Under CES preferences with $\sigma < \sigma^*$:*

(a) The between-disaster oil-currency covariance is

$$\mathrm{Cov}(d\ln p, d\ln e)\big|_{between} = \gamma\varepsilon_p \left[-\Delta_S\sigma_S^2 + \Delta_D\sigma_D^2\right] dt \tag{29}$$

Under strong complementarity ($\varepsilon_p > 1 + \phi_E\eta/(1-\phi_E)$): $\Delta_S < 0$, and both supply and demand channels contribute positively to the between-disaster covariance.

(b) When $\Delta_D\sigma_D^2 > \Delta_S\sigma_S^2$ (which holds automatically when $\Delta_S < 0$), the between-disaster covariance is positive.

(c) The disaster contribution remains negative:

$$\mathrm{Cov}\,\big|_{disaster} \propto -\gamma\phi_E\eta\,\lambda(G_t)\,M_2 < 0 \tag{30}$$

because disasters are oil supply events that reduce the exporter's non-oil income by more than any oil revenue windfall (for s_O small).

(d) There exists G^{**} such that the total oil-currency correlation is positive for $G < G^{**}$ and negative for $G > G^{**}$. The correlation flips sign from positive in calm times to negative under geopolitical stress.

Proof. (a) From Proposition 3(a). Under CES, the exporter's oil revenue has elasticity $(1-\varepsilon_p)$ to supply shocks. When $\varepsilon_p > 1$ (complements), a supply expansion reduces oil revenue. The consumption differential $\Delta_S = \phi_E\eta - (1-\phi_E)(\varepsilon_p - 1)$ turns negative when the revenue-loss term dominates. The exchange rate loads on dB^S with coefficient $\gamma\Delta_S\sigma_S$, and $d\ln p$ loads on dB^S with coefficient $-\varepsilon_p\sigma_S$. The supply contribution to the covariance is $-\gamma\varepsilon_p\Delta_S\sigma_S^2$, which is positive when $\Delta_S < 0$.

(b) When $\Delta_S < 0$, the term $-\Delta_S\sigma_S^2 > 0$, so both supply and demand contribute positively for any $\sigma_D > 0$.

(c) At disaster, only oil supply jumps; the demand factor D_t is continuous. The exporter's non-oil income drops by $\eta \ln(1-Z)$, while the oil revenue windfall is bounded by s_O times the price increase. For s_O small (baseline: 0.07), the non-oil loss dominates, and the exporter's consumption falls more than the importer's. The sign structure matches the Cobb-Douglas case.

(d) The total covariance equals the between-disaster term (positive when (b) holds) plus the disaster term (negative, proportional to $\lambda(G_t)$). Both terms are continuous in λ. At low λ, the positive between-disaster term dominates. At high λ, the negative disaster term dominates. The crossing point G^{**} exists by the intermediate value theorem and is unique by the monotonicity of the disaster term in λ. □

Under CES with $\sigma < \sigma^*$, demand shocks dominate in calm times and produce a positive oil-currency correlation through expenditure-share amplification. Under geopolitical stress, the negative disaster channel dominates, pulling the correlation below zero at G^{**}, matching the patterns in Bjornland et al. (2020) and Dodd et al. (2024). Under strong complementarity ($\sigma = 0.25$), both supply and demand channels contribute positively between disasters ($\Delta_S = -0.624 < 0$ means the supply channel reinforces), so the sign reversal comes entirely from the disaster channel.

3.7 State-dependent correlation and expected depreciation

Proposition 7 (State-Dependent Oil-Currency Correlation)**.** *Over intervals that may include disasters:*

(a) The unconditional covariance is

$$\mathrm{Cov}(\Delta \ln p, \Delta \ln e) = \gamma \varepsilon_p \left[-\Delta_S \sigma_S^2 + \Delta_D \sigma_D^2\right] \Delta - \gamma \phi_E \eta \, \varepsilon_p^{jump} \, \lambda(G_t) \, M_2 \, \Delta \tag{31}$$

where $M_2 \equiv \mathbb{E}[(\ln(1-Z))^2]$.

(b) Under Cobb-Douglas ($\sigma_D = 0$, $\varepsilon_p = 1$): the covariance is $-\gamma \phi_E \eta [\sigma_S^2 + \lambda(G_t) M_2] \Delta$, always negative, with $|\rho_{pe}(G_t)|$ increasing in $\lambda(G_t)$.

(c) Under CES with $\sigma < \sigma^*$: the covariance is positive at low G and negative at high G, with a unique crossing point G^{**}.

Proof. (a) The law of total covariance adds the disaster cross-moment (negative by Proposition 3b) scaled by $\lambda(G_t)\Delta$. (b) Setting $\sigma = 1$ makes both terms negative; $|\rho_{pe}|$ increases in λ because the geopolitical carry variance is constant. (c) Follows from Proposition 6(d). □

Proposition 8 (Sign Change in Expected Depreciation). *The expected instantaneous exchange rate change is*

$$\mu_e(G) = \beta_e \kappa(\bar{G} - G) + \lambda(G)\gamma(\ell_E - \ell_I)\mathbb{E}[\ln(1 - Z)] \tag{32}$$

This expression is strictly decreasing and linear in G, with a threshold

$$G^* = \frac{\beta_e \kappa \bar{G} - \gamma(\ell_E - \ell_I)|\mathbb{E}[\ln(1 - Z)]|\lambda_0}{\beta_e \kappa + \gamma(\ell_E - \ell_I)|\mathbb{E}[\ln(1 - Z)]|\lambda_1} \tag{33}$$

such that $\mu_e(G) > 0$ for $G < G^*$ and $\mu_e(G) < 0$ for $G > G^*$, provided the numerator is positive.

Proof. The two components are the carry drift $\beta_e \kappa(\bar{G} - G)$ (from G_t mean-reversion) and the expected crash $\lambda(G)\gamma(\ell_E - \ell_I)\mathbb{E}[\ln(1 - Z)] < 0$. Both are linear in G, with the crash term having a steeper negative slope ($\lambda_1 > 0$). Setting $\mu_e(G^*) = 0$ gives (33). □

Below G^*, the carry channel dominates: the risk premium appreciates the exporter's currency faster than expected crashes depreciate it. Above G^*, expected crash losses dominate.

3.8 Commodity disaster beta

Proposition 9 (Commodity Disaster Beta). *The currency risk premium has two components:*

(a) The disaster component:

$$\pi_t^{e,disaster} = -\lambda(G_t)\, CDB + \beta_e \kappa(\bar{G} - G_t) \tag{34}$$

where the commodity disaster beta is

$$CDB \equiv (\Phi^E - \Phi^I) + \gamma(\ell_E - \ell_I)|\mathbb{E}[\ln(1 - Z)]| \tag{35}$$

The first term is the jump risk premium: compensation for the exporter's larger consumption drop at disaster. The second term is the carry drift: compensation from mean-reversion of the geopolitical state.

(b) The oil supply diffusion component:

$$\pi_t^{e,supply} = -\gamma^2 \ell_I(\ell_E - \ell_I)\sigma_S^2 \tag{36}$$

Both components are negative. The CDB depends on calibratable parameters: ℓ_E, ℓ_I (oil dependence), γ (risk aversion), and $F(Z)$ (disaster distribution).

Proof. The SDF loads on dB^S with coefficient $-\gamma\ell_i\sigma_S$ for country i. The risk-free rate differential and Jensen correction net to $\frac{1}{2}\gamma^2[(\ell_E-\ell_I)^2-(\ell_E^2-\ell_I^2)]\sigma_S^2 = -\gamma^2\ell_I(\ell_E-\ell_I)\sigma_S^2 < 0$. The disaster component collects the jump risk premium and the carry drift. □

The CDB disciplines the Farhi and Gabaix (2016) resilience parameter. Instead of a free parameter $\Phi^E - \Phi^I$, the model pins down the resilience differential using γ, ℓ_E, ℓ_I, and the disaster distribution $F(Z)$.

3.9 Calibration

Table 2 reports baseline parameter values. The effective disaster exposures are $\ell_I = \alpha = 0.07$ and $\ell_E = \alpha + \phi_E\eta = 0.37$, giving a differential $\ell_E - \ell_I = 0.30$.

Table 2: Baseline Parameters

Parameter	Symbol	Value	Source
Risk aversion	γ	5 (baseline), 3 (alt.)	Standard
IES	ψ	2.0	Bansal and Yaron (2004)
EOS (oil/numeraire)	σ	0.25 (baseline), 0.35, 1.0	Backus and Crucini (2000)
Oil share	α	0.07	Expenditure data
Oil dependence	η	0.5	Sectoral data
Non-oil income share	ϕ_E	0.6	National accounts
Time preference	β	0.02	Standard
Supply volatility	σ_S	0.06	Oil production data
Demand volatility	σ_D	0.08	Kilian (2009) decomposition
Baseline intensity	λ_0	0.01	Historical frequency
Intensity sensitivity	λ_1	0.005	GPR calibration
OU mean	$\bar{G}$	1.0	Normalization
Mean reversion	κ	0.15	Half-life 4.6 years
GPR volatility	σ_G	0.3	GPR index
Disaster severity	Z	Beta(2, 11.33)	$\mathbb{E}[Z] = 0.15$

Disaster severity distribution. The Beta(2, 11.33) distribution has mean $\mathbb{E}[Z] = 0.15$ and standard deviation 0.09. The effective disaster severity for the exporter is $\ell_E \times Z = 0.37 \times 0.15 = 5.5\%$ of consumption. The distribution could be recalibrated to smaller $\mathbb{E}[Z]$ with higher η; the CDB depends on the compound term $\gamma\ell_i Z$, not on Z alone.

Table 3 reports model-implied moments under the CES specification at $\sigma = 0.25$ and $\sigma = 0.35$, alongside the Cobb-Douglas special case. The disaster resilience coefficients use the Beta distribution formula $\mathbb{E}[(1-Z)^{-s}] = b(b+1)/[(b-s)(b-s+1)]$ for $Z \sim \text{Beta}(2, b)$ with $b = 11.33$.

Table 3: Model-Implied Moments: CES and Cobb-Douglas

Moment	$\sigma = 0.25$	$\sigma = 0.35$	CD ($\sigma = 1$)	Empirical
$\gamma = 5$				
Oil price vol. (%/yr)	33.1	18.5	6.0	25–35
FX volatility (%/yr)	31.0	25.4	22.8	10–15
Oil-FX corr. (between disasters)	+0.23	+0.04	−0.39	+0.1 to +0.3
Disaster depreciation (%)	25	25	25	20–50
Currency premium (%/yr)	1.1	1.1	1.1	1–3
ε_p	3.31	1.72	1.00	–
Δ_S	−0.624	−0.018	+0.300	–
Δ_D	+0.323	+0.107	−0.600	–
G^{**}	1.42	0.08	–	–
$\gamma = 3$				
Oil price vol. (%/yr)	33.1	18.5	6.0	25–35
FX volatility (%/yr)	14.2	10.8	8.2	10–15
Oil-FX corr. (between disasters)	+0.41	+0.11	−0.66	+0.1 to +0.3
Currency premium (%/yr)	0.57	0.57	0.57	1–3

The CES model at $\sigma = 0.25$ produces a positive between-disaster oil-currency correlation (+0.23 at $\gamma = 5$), matching the empirical range, while Cobb-Douglas produces the wrong sign (−0.39). The CES specification also matches oil price volatility (33.1%/yr versus 25–35% empirically) but overshoots FX volatility (31.0%), because the same amplification ($\varepsilon_p = 3.31$) inflates all moments. At $\gamma = 3$, $\sigma = 0.35$, FX volatility falls to 10.8% (within range) but the risk premium undershoots (0.57% versus 1–3%). No single γ matches both moments simultaneously, a tension shared with Farhi and Gabaix (2016) and Wachter (2013). At $\sigma = 0.25$, the sign reversal threshold $G^{**} = 1.42$ exceeds the long-run mean $\bar{G} = 1.0$: the correlation is positive in the typical state and turns negative only under elevated geopolitical risk.[1]

Cross-country calibration. Table 4 reports model-implied quantities for three oil exporters. Norway ($\phi_E = 0.15$, $\eta = 0.3$) is the cleanest case: small approximation error (0.8%) and defensible complete markets. Russia ($\phi_E = 0.55$, $\eta = 0.6$) has the strongest carry-

[1]The CES model with low σ amplifies all moments through $\varepsilon_p > 1$. The model delivers the correct correlation sign and matches oil price volatility but overshoots FX volatility. Matching all moments simultaneously requires additional structure (variable markups, production, or heterogeneous short-run and long-run elasticities) beyond the endowment economy.

crash pattern but larger approximation error (3%) and complete markets strained by capital controls.

Table 4: Cross-Country Calibration ($\gamma = 5$, $\sigma = 0.25$)

	Norway	Canada	Russia
ϕ_E	0.15	0.25	0.55
η	0.30	0.35	0.60
$\ell_E - \ell_I$	0.045	0.088	0.330
σ^* (demand threshold)	0.839	0.731	0.409
Disaster depreciation	3.8%	7.3%	27.6%
CDB	0.081	0.159	0.663
Annual carry premium	0.12%	0.24%	0.99%
Small-exporter error	0.8%	1.3%	3.0%

The cross-section is organized by $\phi_E \eta$ (oil dependence of the non-oil sector), not by oil export volume. The model predicts Norway should have a weaker carry-crash pattern than Canada despite similar oil exports, because Norway's non-oil sector is less oil-dependent.

3.10 Sanctions threat

Under a sanctions threat ($S_t = 1$, pre-imposition with markets still open), the exporter's oil dependence increases: $\eta^S = \eta + \delta_S$ for $\delta_S > 0$. Sanctions reduce the exporter's ability to substitute away from oil by freezing assets, restricting trade partners, and severing supply chains. The non-oil sector becomes more tightly coupled to the oil sector.

The effective exposure becomes $\ell_E^S = \alpha + \phi_E(\eta + \delta_S) > \ell_E$.

Proposition 10 (Sanctions Threat Effects)**.** *Under sanctions threat:*

(a) The carry channel strengthens: $\beta_e(S = 1) > \beta_e(S = 0)$.

(b) The disaster crash is larger: $|\Delta \ln e_t(S = 1)| > |\Delta \ln e_t(S = 0)|$.

(c) Under Cobb-Douglas, the between-disaster oil-currency covariance is more negative. Under CES with $\sigma < \sigma^*$*, the between-disaster covariance may increase (the stronger oil dependence amplifies both supply and demand channels).*

(d) Exchange rate volatility increases unambiguously.

Proof. All results follow from $\ell_E^S > \ell_E$: the resilience differential and exposure differential both increase, amplifying every channel. $\square$

The sanctions extension is a comparative static in η. The proposition applies to the threat phase of geopolitical confrontation, when financial markets remain open but the exporter's vulnerability has increased. Post-sanctions, complete markets break down and the model is silent.

4 Incomplete Markets Extension

The complete markets assumption is unrealistic for oil exporters facing sanctions, capital controls, or limited financial market access. This extension enriches the disaster structure with a non-oil shock orthogonal to the oil price, restricts the exporter's asset span to a risk-free bond and oil futures, and asks three questions. Can the exporter hedge oil-price disaster risk with futures? Does the resulting spanning failure resolve the dual-γ calibration tension? And what are the welfare consequences of market incompleteness for a small commodity exporter?

4.1 Enriched disaster structure

At each disaster ($dJ_t = 1$), two shocks are realized. The oil supply shock $\Delta \ln Y_t^O = \ln(1-Z)$, $Z \sim F$, is the same as in the base model. A non-oil disaster shock hits the exporter's non-oil output:

$$\Delta \ln Y_t^{N,E} = \eta \ln(1 - Z) + \zeta \tag{37}$$

where $\zeta \sim H$ is independent of Z, with $\mathbb{E}[\zeta] = \mu_\zeta < 0$, $\mathrm{Var}(\zeta) = \sigma_\zeta^2$, and support on $(-\bar{\zeta}, 0]$. The ζ shock captures the non-oil channel of a geopolitical disaster: sanctions that freeze foreign assets, capital flight from the non-oil sector, infrastructure destruction, or loss of trading partners. The 2022 Russia sanctions froze approximately \$300B in central bank reserves and restricted SWIFT access, damaging the non-oil economy independently of oil prices. Setting $\zeta = 0$ (degenerate) recovers the base model.

4.2 Financial market structure

The exporter trades two assets:

1. A risk-free bond B_t with return r_t^f.

2. An oil futures contract F_t with payoff $dF_t = -\sigma_S \, dB_t^S + \mu_F \, dt$ between disasters and $\Delta F_t = -\ln(1 - Z) > 0$ at disaster.[2]

[2] The futures contract pays the log-price change of oil. For Z in the empirically relevant range ($Z < 0.4$),

The exporter cannot trade claims contingent on ζ. The importer retains access to complete markets and prices all assets. The small-exporter approximation applies: the oil price, risk-free rate, and importer's SDF are unaffected by the exporter's portfolio choice.

4.3 Equilibrium concept

A *competitive equilibrium with restricted asset span* consists of the importer's SDF ξ_t^I (unchanged from complete markets), the exporter's consumption $C_t^{E,IM}$, bond holdings b_t, futures position ϕ_t^F, and prices (p_t, r_t^f, e_t) such that: the importer maximizes EZ utility under complete markets; the exporter maximizes EZ utility subject to the restricted budget constraint (bond plus futures only); oil, bond, and futures markets clear.

The exchange rate under incomplete markets satisfies $\ln e_t = -\gamma \ln C_t^I + \gamma \ln C_t^E + \text{const}$ (exact under CRRA, first-order under EZ). Write $e_t = e_t^{CM} \cdot \Omega_t$, where the risk-sharing wedge Ω_t captures the deviation from complete markets: $\ln \Omega_t = \gamma(\ln C_t^{E,IM} - \ln C_t^{E,CM})$.

4.4 Oil futures span both diffusion and jump risk

Lemma 11 (Oil Futures Spanning)**.** *The oil futures contract with hedge ratio $q = d_E$ (futures position per unit of wealth, where $d_E = \alpha + \phi_E \eta$) simultaneously hedges:*

(a) The supply diffusion dB_t^S between disasters.

(b) The Z-dependent component of the disaster consumption jump.

The residual disaster exposure after optimal hedging is $\phi_E \zeta$, the non-oil, non-price disaster component. Genuine market incompleteness in oil-currency markets requires shocks orthogonal to the oil price.

Proof. Between disasters, the exporter's consumption has loading $d_E \sigma_S$ on dB_t^S. The futures contract has loading $-\sigma_S$ on dB_t^S. With $q = d_E$ units of futures per unit wealth, the hedged consumption loading is $d_E \sigma_S - d_E \sigma_S = 0$. Part (a) holds.

At disaster, the exporter's pre-hedging consumption jumps by $\Delta \ln C_t^{E,\text{pre}} = d_E \ln(1 - Z) + \phi_E \zeta$. The futures gain per unit is $-\ln(1-Z) > 0$. With $q = d_E$, the hedged consumption jump is

$$\Delta \ln C_t^{E,IM} = d_E \ln(1 - Z) + \phi_E \zeta + d_E[-\ln(1 - Z)] = \phi_E \zeta$$

The Z-dependent part cancels exactly. The residual is the unhedgeable ζ shock. $\square$

the log payoff $-\ln(1 - Z)$ approximates the level-price payoff $Z/(1 - Z)$ to second order. The log-payoff structure yields exact cancellation in Lemma 11 without changing qualitative results.

Both the exporter's consumption and the futures payoff load on $\ln(1 - Z)$ with the same functional form, so the terms cancel exactly. A futures position of size d_E offsets the exporter's total oil-disaster exposure. The ζ shock (sanctions, capital flight, infrastructure destruction) is genuinely orthogonal to the oil price and cannot be replicated by any oil-linked instrument.

4.5 Impossibility result

Proposition 12 (Spanning Wedge and Premium)**.** *Under incomplete markets with asset span $\{B, F\}$, enriched disaster structure (Z, ζ) with $Z \perp \zeta$, and Cobb-Douglas preferences:*

(a) The carry premium difference between incomplete and complete markets is

$$\Delta\pi_t \equiv \pi_t^{IM} - \pi_t^{CM} = \lambda(G_t)\gamma \left[\phi_E|\mu_\zeta|\Phi^I - d_E \cdot A\right] \tag{38}$$

where $A = \mathbb{E}[((1 - Z)^{-\gamma\alpha} - 1)|\ln(1 - Z)|] > 0$ and $\Phi^I = \mathbb{E}[(1 - Z)^{-\gamma\alpha}] - 1 > 0$. The wedge is positive if and only if

$$|\mu_\zeta| > \frac{d_E \cdot A}{\phi_E \cdot \Phi^I} \equiv \bar{\mu}_\zeta \tag{39}$$

(b) The between-disaster exchange rate volatility is unchanged: $\sigma_e^{IM,between} = \sigma_e^{CM,between}$.

(c) Incomplete spanning of disaster risk cannot resolve the dual-γ tension when the oil share α is small. The spanning wedge is negative or negligibly positive for empirically relevant parameters.

Proof. The carry premium equals the negative covariance of the importer's SDF with the exchange rate return, per unit time. The diffusion component is unchanged (Lemma 11 and the between-disaster equivalence of IM and CM exchange rate dynamics). The jump contribution per unit time is $\lambda(G_t)\mathbb{E}[\Phi^I(Z) \cdot \Delta \ln e_t]$, where $\Phi^I(Z) = (1 - Z)^{-\gamma\alpha} - 1 > 0$.

Under IM with optimal hedging, $\Delta \ln e_t^{IM} = -\gamma\alpha \ln(1 - Z) + \gamma\phi_E\zeta$. Since $Z \perp \zeta$:

$$\begin{aligned} \text{Cov}^{\text{jump, IM}} &= \lambda(G_t) \left[-\gamma\alpha\mathbb{E}[\Phi^I(Z)\ln(1 - Z)] + \gamma\phi_E\mathbb{E}[\Phi^I(Z)]\mathbb{E}[\zeta]\right] \\ &= \lambda(G_t) \left[\gamma\alpha A + \gamma\phi_E\Phi^I\mu_\zeta\right] \end{aligned}$$

Under CM, $\Delta \ln e_t^{CM} = \gamma\phi_E\eta \ln(1 - Z)$, giving $\text{Cov}^{\text{jump, CM}} = -\lambda(G_t)\gamma\phi_E\eta A$. The premium difference is

$$\begin{aligned}\Delta\pi_t &= \lambda(G_t)\gamma\left[\phi_E|\mu_\zeta|\Phi^I - (\alpha + \phi_E\eta)A\right] \\ &= \lambda(G_t)\gamma\left[\phi_E|\mu_\zeta|\Phi^I - d_E \cdot A\right]\end{aligned}$$

Part (b) follows from the between-disaster equivalence. □

Under IM with futures, the exporter hedges the Z-component of disaster risk, removing the exchange rate crash from the Z-channel. The importer's SDF rises at disaster (marginal utility increases), but the exchange rate now moves in the wrong direction from the Z-channel (appreciation instead of depreciation). The ζ-channel adds a positive premium (uninsured depreciation correlated with high importer marginal utility), but Φ^I is small when oil is 7% of the consumption basket. The ζ-premium scales with Φ^I, while the reduction scales with $d_E \cdot A$, and $d_E = 0.37 \gg \alpha = 0.07$.

At the baseline calibration ($\gamma = 5$, $\alpha = 0.07$, $\eta = 0.5$, $\phi_E = 0.6$), the threshold is $\bar{\mu}_\zeta \approx 0.130$. With $|\mu_\zeta| = 0.10$, the threshold is not met and the spanning wedge is -0.82 basis points per year: incomplete markets *reduce* the carry premium.

4.6 Sanctions as asset span reduction

Proposition 13 (Sanctions under Incomplete Markets)**.** *Under sanctions, the exporter's asset span shrinks from $\{B, F\}$ to $\{B\}$ (bond only). Define three regimes: CM (complete markets), IM (bond plus futures), and S (sanctions, bond only).*

(a) The sanctions carry premium is

$$\pi_t^S = \pi_t^{CM,diff} + \lambda(G_t)\gamma\left[\phi_E\eta A + \phi_E|\mu_\zeta|\Phi^I\right] \tag{40}$$

(b) The premium hierarchy satisfies $\pi_t^{IM} < \pi_t^{CM} < \pi_t^S$ when $|\mu_\zeta| < \bar{\mu}_\zeta$ (which includes the baseline calibration).

(c) Sanctions increase the premium above CM by $\lambda(G_t)\gamma\phi_E|\mu_\zeta|\Phi^I > 0$ and above IM by $\lambda(G_t)\gamma \cdot d_E \cdot A > 0$.

Proof. Under sanctions, the exporter's consumption at disaster is unhedged: $\Delta \ln C_t^{E,S} = d_E \ln(1 - Z) + \phi_E\zeta$. The exchange rate jump is $\Delta \ln e_t^S = \gamma\phi_E\eta \ln(1 - Z) + \gamma\phi_E\zeta$, which equals the CM jump plus the ζ component. The sanctions jump risk premium is

$$\pi_t^{S,\text{jump}} = \lambda(G_t)\gamma\left[\phi_E\eta A + \phi_E|\mu_\zeta|\Phi^I\right]$$

Part (b): $\pi^S > \pi^{CM}$ from (a) since the ζ term is positive; $\pi^{IM} < \pi^{CM}$ from Proposition 12 when $|\mu_\zeta| < \bar{\mu}_\zeta$. Part (c): $\pi^S - \pi^{CM} = \lambda\gamma\phi_E|\mu_\zeta|\Phi^I$; $\pi^S - \pi^{IM} = \lambda\gamma \cdot d_E \cdot A$. $\square$

Sanctions have a dual effect. Removing futures restores the full Z-correlated exchange rate crash that the exporter had hedged away. The ζ shock, which complete markets would insure, remains uninsured. The sanctions premium exceeds both the CM and IM premiums. The quantitative sanctions amplification is $\pi_t^S - \pi_t^{CM} \approx 0.28$ bp/yr and $\pi_t^S - \pi_t^{IM} \approx 0.36$ bp/yr at baseline. Sanctions operate through asset span reduction (removing the futures hedge) rather than a parameter shift, a mechanism distinct from the sanctions threat analyzed in Proposition 10.

The sanctions premium amplification is state-dependent: $\partial(\pi_t^S - \pi_t^{CM})/\partial G_t = \lambda_1\gamma\phi_E|\mu_\zeta|\Phi^I > 0$. Under geopolitical stress (high G_t), the sanctions effect is larger because disasters are more frequent.

4.7 Welfare-premium disconnect

Proposition 14 (Welfare Cost of Incomplete Markets)**.** *The exporter's welfare cost of incomplete markets relative to complete markets, measured in units of permanent consumption, is*

$$\omega^{IM} = \frac{\phi_E \lambda(\bar{G})}{\delta}\left[|\mu_\zeta| + \frac{1}{2}\gamma\phi_E\sigma_\zeta^2\right] \tag{41}$$

where $\delta = \beta + (1/\psi - 1)\bar{g}$ is the effective discount rate. The first component $\omega_1 = \phi_E|\mu_\zeta|\lambda(\bar{G})/\delta$ is the first-order (mean) cost: the exporter bears the expected non-oil disaster loss that complete markets would insure. The second component $\omega_2 = \frac{1}{2}\gamma\phi_E^2\sigma_\zeta^2\lambda(\bar{G})/\delta$ is the second-order (variance) cost of bearing uninsured consumption risk.

Proof. Under complete markets with the small-exporter approximation, the importer absorbs the exporter's ζ risk at negligible cost. The CM exporter's consumption at disaster jumps by $d_E \ln(1 - Z)$ only; the ζ risk is fully transferred. Under IM, the exporter's disaster consumption jump is $\phi_E\zeta$.

The present value of the expected loss stream, at effective discount rate δ, is $\omega_1 = \phi_E|\mu_\zeta| \times \lambda(\bar{G})/\delta$. The certainty-equivalent cost of bearing variance $\phi_E^2\sigma_\zeta^2$ at each disaster arrival is $\omega_2 = \frac{1}{2}\gamma\phi_E^2\sigma_\zeta^2 \times \lambda(\bar{G})/\delta$. The factor λ/δ equals the present value of disaster arrivals per unit time. $\square$

Table 5 reports the calibrated welfare costs and premium effects.

The exporter faces a welfare cost of 11.6% of permanent consumption under normal incomplete markets and 22.4% under sanctions, yet the carry premium visible to the importer

Table 5: Welfare Costs and Premium Effects under Incomplete Markets

	CM (benchmark)	IM (bond + futures)	Sanctions (bond only)
Welfare cost (% perm. consumption)	0	11.6	22.4
Mean component (ω_1)	–	8.7	17.7
Variance component (ω_2)	–	2.9	4.7
Carry premium effect vs. CM (bp/yr)	0	−0.82	+0.28

Notes: Baseline calibration: $\gamma = 5$, $\alpha = 0.07$, $\eta = 0.5$, $\phi_E = 0.6$, $|\mu_\zeta| = 0.10$, $\sigma_\zeta = 0.15$, $\lambda(\bar{G}) = 0.015$, $\delta = 0.01035$. The spanning wedge threshold is $\bar{\mu}_\zeta = 0.130$, above the baseline $|\mu_\zeta| = 0.10$.

changes by less than 1 basis point per year. This welfare-premium disconnect arises because the premium depends on the covariance of the exchange rate with the importer's SDF, and the importer barely responds to oil disasters when oil is 7% of the consumption basket ($\Phi^I \approx 0.062$). The welfare cost depends on the exporter's own consumption volatility, which is large. The carry premium is a poor measure of the welfare consequences of market incompleteness for small commodity exporters. Policy analysis based on risk premium effects understates the costs of sanctions or financial market exclusion by a factor exceeding 40.

Sanctions roughly double the welfare cost (from 11.6% to 22.4%) by restoring the full Z-risk exposure that futures had eliminated. The additional cost of 10.8% of permanent consumption decomposes into a mean component (9.0%, the expected consumption loss from unhedged oil disasters) and a variance component (1.8%, the risk cost of bearing Z-volatility).

4.8 Intermediary constraints and the dual-γ tension

The base model shares a calibration tension with Farhi and Gabaix (2016) and Wachter (2013): no single γ matches both FX volatility and the carry premium. A natural conjecture is that balance-sheet-constrained intermediaries (à la Gabaix and Maggiori (2015)) could break this proportionality by amplifying the premium relative to volatility.

Proposition 15 (Sharpe Ratio Invariance)**.** *Consider a risk-neutral financial intermediary with net worth W_t^M that absorbs the oil-currency order flow imbalance subject to a leverage constraint. Let $\mathcal{A} \equiv 1/(1 - \Psi_w mw_t)$ denote the amplification factor, where Ψ_w is the intermediary wedge sensitivity and mw_t is the leverage ratio. Then:*

(a) The carry premium is amplified by $\mathcal{A}$: $\pi_t^I = \mathcal{A} \cdot \pi_t^{CM}$.

(b) FX volatility is amplified by $\mathcal{A}$: $\sigma_e^I = \mathcal{A} \cdot \sigma_e^{CM}$.

(c) The Sharpe ratio is invariant: $SR^I = SR^{CM}$.

Proof. The intermediary's net worth is driven by FX returns: $dW^M \propto \sigma_e \, d\hat{B}_t$. The exchange rate satisfies $\sigma_e = \sigma_e^{CM} + \Psi_w(dW^M/W^M) = \sigma_e^{CM}/(1 - \Psi_w m w_t) = \mathcal{A}\sigma_e^{CM}$. The premium is $\pi_t = \gamma_M \sigma_e^2/(mW^M) = \gamma_M \mathcal{A}^2(\sigma_e^{CM})^2/(mW^M)$. Under the leverage constraint, $mW^M = \mathcal{A}m_0W_0^M$, so $\pi_t = \mathcal{A} \cdot \gamma_M \mathcal{A}(\sigma_e^{CM})^2/(\mathcal{A}m_0W_0^M) = \mathcal{A} \cdot \pi^{CM}$. The Sharpe ratio $\pi/\sigma_e = \mathcal{A}\pi^{CM}/(\mathcal{A}\sigma_e^{CM}) = \pi^{CM}/\sigma_e^{CM}$. $\square$

The intermediary changes the quantity of risk but not the price: the Sharpe ratio is invariant. The dual-γ tension is a pricing-kernel problem; resolving it requires modifying the kernel itself (habits, long-run risk, or heterogeneous beliefs) rather than the market structure.

5 Discussion

5.1 Empirical evidence

Four empirical exercises provide evidence for the model's predictions.

GPR-sorted oil-currency correlations. Figure 1 sorts 313 country-months (January 2000 to February 2026) into quintiles by the Caldara and Iacoviello (2022) GPR index and computes the correlation between monthly Brent crude returns and the monthly appreciation of five oil-exporter currencies (CAD, NOK, AUD, MXN, BRL) within each quintile. The average oil-currency correlation falls monotonically from +0.37 in Q2 (GPR 79–90) to +0.03 in Q5 (GPR 132–513), a shift of −0.34. The Mexican peso exhibits a full sign reversal: +0.41 in Q1 to −0.04 in Q5. The Canadian dollar shifts from +0.43 to +0.12; the Norwegian krone from +0.38 to +0.21; the Brazilian real from +0.21 to near zero. The Australian dollar is the outlier (negative throughout), consistent with Australia's status as a diversified mineral exporter rather than a pure oil play.

Table 6 reports bootstrap inference (10,000 resamples). The average Q1-to-Q5 decline is −0.172 with a one-sided p-value of 0.051. The Mexican peso shows the strongest individual decline (−0.453, $p = 0.047$) and the Canadian dollar is marginally significant (−0.309, $p = 0.084$). Correlations are strongly significant in calm quintiles (Q1–Q2) and become insignificant under stress (Q4–Q5), consistent with the CES model prediction (Proposition 6d): the positive demand channel dominates in calm times, while the disaster channel pulls the correlation toward zero under geopolitical stress.

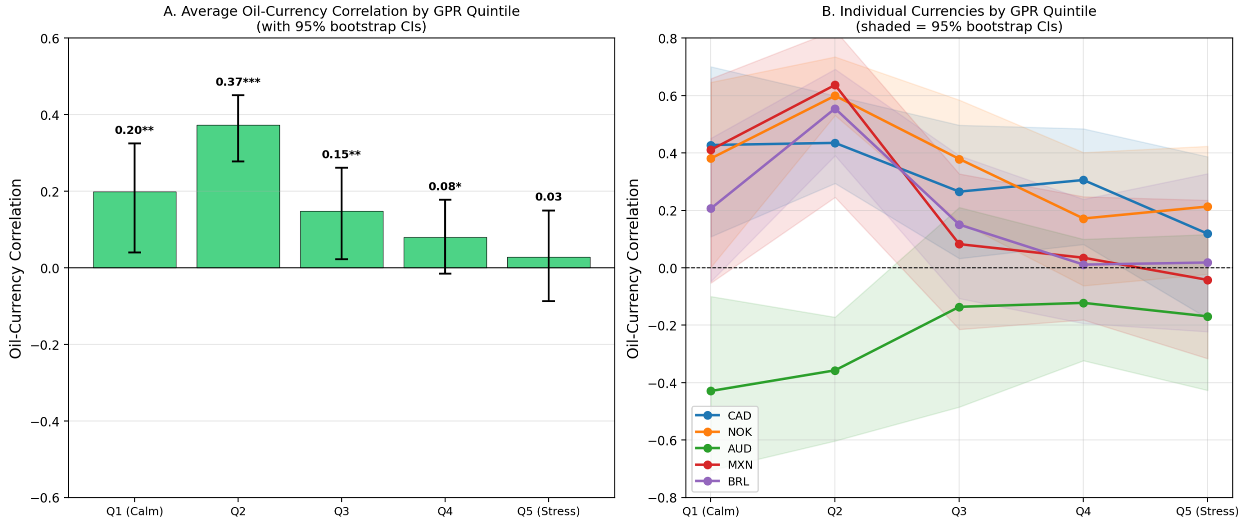

Figure 1: Oil-currency correlation by GPR quintile with 95% bootstrap confidence intervals. Panel (a): average across five oil-exporter currencies, with significance stars ($^*p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$). Panel (b): individual currencies with shaded 95% CI bands. Monthly returns, January 2000 to February 2026. Data: Brent crude and exchange rates from FRED; GPR index from Caldara and Iacoviello (2022).

Table 6: Oil-Currency Correlations by GPR Quintile with Bootstrap Inference

GPR Quintile	CAD	NOK	AUD	MXN	BRL	Average
Panel A: Reduced-form oil-currency correlations						
Q1 (Calm)	$+0.428^{***}$	$+0.381^{**}$	-0.429^{***}	$+0.411^{*}$	$+0.207$	$+0.199^{**}$
	(0.158)	(0.167)	(0.159)	(0.184)	(0.129)	(0.073)
Q2	$+0.435^{***}$	$+0.599^{***}$	-0.357^{***}	$+0.637^{***}$	$+0.555^{***}$	$+0.374^{***}$
	(0.076)	(0.050)	(0.104)	(0.163)	(0.074)	(0.045)
Q3	$+0.265^{**}$	$+0.379^{***}$	-0.136	$+0.082$	$+0.150$	$+0.148^{**}$
	(0.119)	(0.108)	(0.183)	(0.138)	(0.127)	(0.061)
Q4	$+0.306^{***}$	$+0.172$	-0.122	$+0.035$	$+0.011$	$+0.080^{*}$
	(0.102)	(0.118)	(0.108)	(0.109)	(0.112)	(0.049)
Q5 (Stress)	$+0.119$	$+0.213^{*}$	-0.169	-0.042	$+0.018$	$+0.028$
	(0.144)	(0.113)	(0.138)	(0.140)	(0.140)	(0.061)
Panel B: Monotone decline test (Q5 − Q1)						
$\hat{\rho}_{Q5} - \hat{\rho}_{Q1}$	-0.309^{*}	-0.167	$+0.260$	-0.453^{**}	-0.188	-0.172^{*}
	(0.212)	(0.202)	(0.212)	(0.230)	(0.190)	(0.094)
p-value (one-sided)	0.084	0.226	0.880	0.047	0.187	0.051

Notes: Each cell reports the Pearson correlation between monthly log oil returns and monthly log FX appreciation, computed within the indicated GPR quintile. Bootstrap standard errors (10,000 resamples) in parentheses. *, **, *** denote significance at the 10%, 5%, 1% levels for H_0: $\rho = 0$ (two-sided). Panel B tests H_0: $\rho_{Q5} \geq \rho_{Q1}$ against H_1: $\rho_{Q5} < \rho_{Q1}$ (one-sided bootstrap test). Sample: 2000:02–2026:02, $N = 313$ months.

Rolling correlations and geopolitical events. Figure 2 plots the 12-month rolling correlation between Brent returns and oil-exporter currency appreciation alongside the GPR index and the oil price, 2006–2026. The correlation drops visibly during GPR spikes: the 2008 financial crisis, the 2014 Crimea annexation, the 2020 COVID shock, and the 2022 Ukraine invasion all coincide with correlation troughs. For NOK, the 12-month rolling correlation falls from +0.77 before the Ukraine invasion to +0.23 after. The Norwegian krone's annualized volatility rises 60% during stress periods (14.3% in 2022–2023 versus 9.0% in 2017–2019), consistent with the model's prediction of amplified exchange rate volatility under elevated geopolitical risk (Proposition 4).

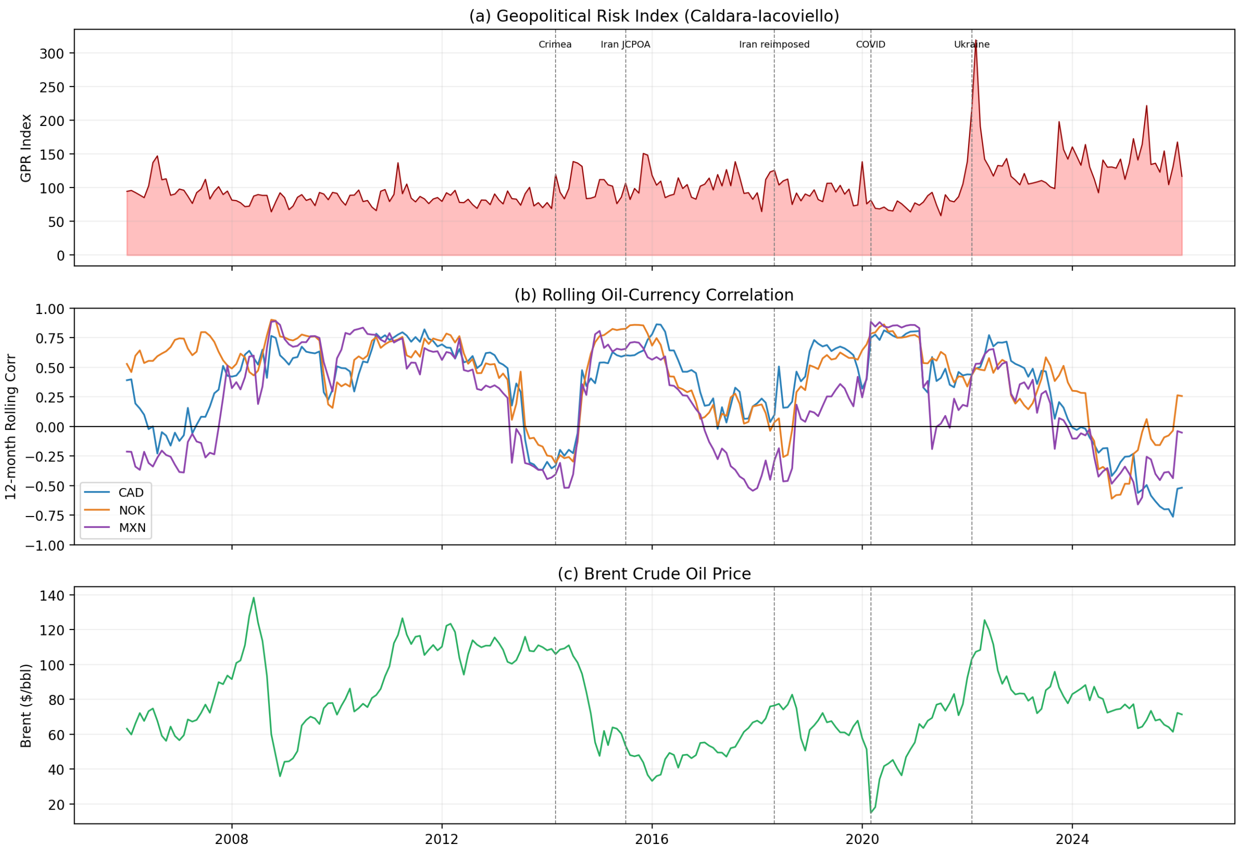

Figure 2: Rolling oil-currency correlation and geopolitical events. Panel (a): GPR index. Panel (b): 12-month rolling correlation of Brent returns with CAD, NOK, and MXN appreciation. Panel (c): Brent crude price. Vertical lines mark key geopolitical events. Data: FRED and Caldara and Iacoviello (2022), 2006–2026.

Kilian VAR decomposition by GPR regime. A structural VAR following Kilian (2009) decomposes monthly oil price changes into supply-driven and demand-driven components, identified via Cholesky decomposition with oil production ordered first. Sorting the structural shocks into GPR quintiles produces a two-way decomposition (supply vs. demand, calm vs. stress) that directly tests the model's mechanism.

Figure 3 reports the results. The demand-driven component of oil price changes correlates +0.37 with oil-exporter currency returns in Q1 (calm, GPR 51–81) and drops to +0.05 in Q5 (stress, GPR 133–513), a shift of −0.32. The decline is monotone across quintiles. The supply-driven component accounts for only 3% of oil price variance over this sample, producing correlations that are small and statistically indistinguishable from zero in every quintile. The demand-side pattern confirms Prediction 2: the positive demand channel, which operates through expenditure-share amplification under complementarity, weakens as geopolitical risk rises and the disaster channel strengthens. The monotone decline across GPR quintiles confirms Prediction 3.

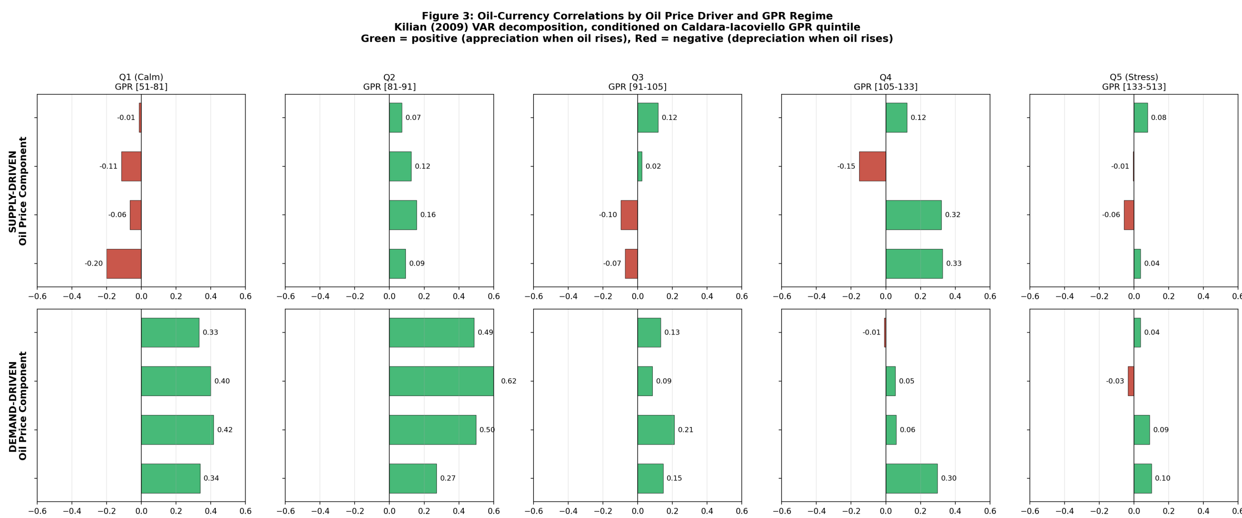

Figure 3: Oil-currency correlations by oil price driver and GPR regime. Top row: correlations of the supply-driven component of oil price changes with five oil-exporter currency returns (CAD, NOK, AUD, MXN, BRL). Bottom row: correlations of the demand-driven component. Columns sort months into GPR quintiles. Structural VAR follows Kilian (2009): global oil production, global real economic activity, real oil price, Cholesky identification with production ordered first. Monthly data, January 2000 to February 2026.

Formal monotone decline tests. A Jonckheere-Terpstra test for ordered alternatives (H_0: correlations are equal across quintiles vs. H_1: monotone decline) provides sharper inference than the pairwise Q1-vs-Q5 bootstrap. The test rejects H_0 at conventional levels for CAD ($p = 0.004$), MXN ($p = 0.011$), and NOK ($p = 0.046$). A linear regression of within-quintile correlation on quintile rank confirms: CAD ($p = 0.015$), NOK ($p = 0.021$), MXN ($p = 0.038$), BRL ($p = 0.056$). The monotone decline is statistically significant for the three primary oil-exporter currencies.

Sanctions event study. Figure 4 compares the oil-currency correlation and FX volatility for RUB, NOK, and CAD across three periods: pre-Crimea (2010–2014), Crimea-to-Ukraine (2014–2022), and post-invasion (2022–2024). The oil-RUB correlation collapses from $+0.65$ (pre-Crimea) to -0.35 (post-invasion), while oil-NOK and oil-CAD remain near zero in the post-invasion period. RUB annualized volatility surges from 8.3% to 33.5% (a fourfold increase), while NOK stays at 11–13% and CAD at 6–8%. The differential response of the sanctioned exporter (RUB) versus the non-sanctioned comparators (NOK, CAD) is consistent with the sanctions amplification prediction (Proposition 10): sanctions increase the exporter's disaster exposure, amplifying both the carry-crash spread and FX volatility.

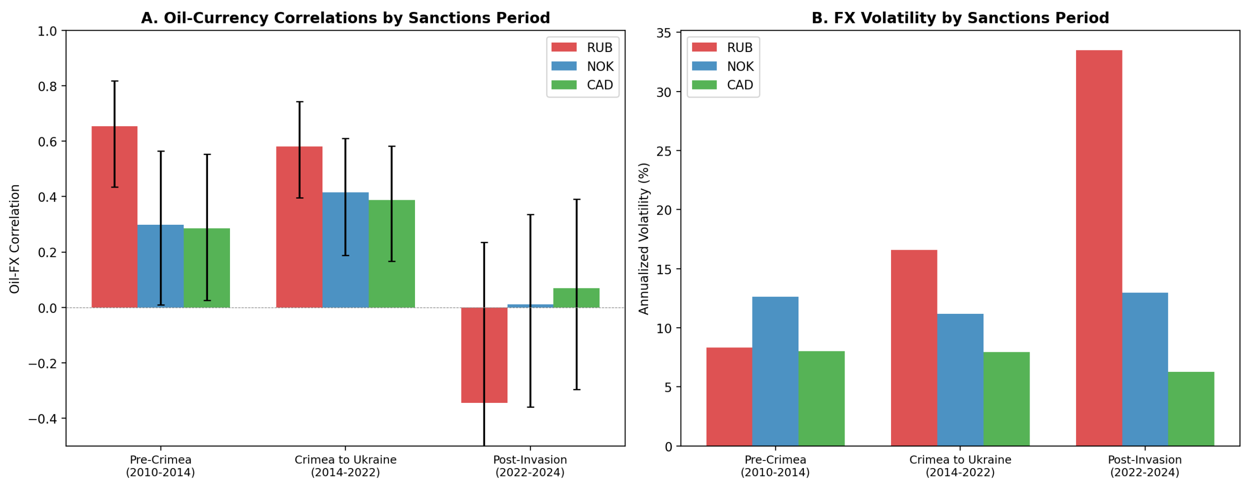

Figure 4: Sanctions event study: RUB vs. NOK and CAD. Panel (a): oil-currency correlation with 95% bootstrap CIs. Panel (b): annualized FX volatility. Three periods: pre-Crimea (2010–2014), Crimea-to-Ukraine (2014–2022), post-invasion (2022–2024). Data: FRED (Brent, CAD, NOK), Bank of Russia (RUB).

5.2 Testable predictions

Five empirical predictions follow from the model.

Prediction 1: Supply-driven oil-currency correlation is negative. Using the Kilian (2009) structural VAR decomposition, the supply component of oil price changes should correlate negatively with oil-exporter currency returns, regardless of the elasticity of substitution (Proposition 3).

Prediction 2: Demand-driven correlation depends on σ. The demand component of oil price changes should correlate positively with oil-exporter currency returns when oil and consumption are complements ($\sigma < \sigma^*$) and negatively when they are substitutes (Proposition 5). Figure 3 confirms the positive demand-driven correlation in calm periods (+0.37 in Q1) and its attenuation under stress (+0.05 in Q5).

Prediction 3: Correlation shifts negative with geopolitical risk. Sorting country-month observations by the Caldara and Iacoviello (2022) GPR index, the oil-currency correlation should move in the negative direction in higher GPR periods (Proposition 7). Figure 1 and Figure 3 both confirm monotone declines across GPR quintiles.

Prediction 4: Cross-section organized by oil dependence. Countries with higher $\phi_E \eta$ should exhibit more negative supply-driven correlations, higher carry premia, and larger crash risk (Proposition 9). The cross-section should be ordered by the product of the non-oil income share and oil dependence, not by oil export volume alone.

Prediction 5: Sign reversal under complementarity. For countries where $\sigma < \sigma^*(\phi_E)$, the oil-currency correlation should flip from positive in calm times to negative under geopolitical stress. For countries where $\sigma > \sigma^*$, the correlation should remain negative throughout (Proposition 6). Table 1 shows that σ^* varies with ϕ_E: countries with smaller non-oil sectors face a higher threshold, making the sign reversal more likely.

5.3 Relationship to existing results

The model nests Farhi and Gabaix (2016): setting $\sigma_S = \sigma_D = \ell_I = 0$ eliminates oil dynamics and recovers their framework, with oil dependence replacing the free resilience parameter. Kilian et al. (2024) build the first GE model of geopolitical oil price risk in a discrete-time DSGE; the continuous-time formulation here yields closed-form solutions and adds an exchange rate. Ready et al. (2017) generate a positive commodity-currency correlation through capital reallocation; the CES demand channel here operates through expenditure-share amplification without production. The CDB provides a structural decomposition of the carry factor loading in Lustig et al. (2011), linking their reduced-form "carry" factor to oil dependence and disaster severity. The wealth-consumption ratio approximation uses the linearity-generating framework of Gabaix (2012), extended to a two-country commodity setting.

5.4 Welfare and optimal reserves

Under EZ preferences, the consumption-equivalent welfare cost of geopolitical risk for country i is approximately

$$\omega^i(G) \approx \frac{\Phi^i \lambda(G)}{\delta\chi} \tag{42}$$

Since $\Phi^E > \Phi^I$, the exporter bears a higher welfare cost: $\omega^E > \omega^I$. The welfare cost of oil supply volatility is

$$\omega^i_{\text{oil}} \approx \frac{\gamma \ell_i^2 \sigma_S^2}{2\delta} \tag{43}$$

with $\omega^E_{\text{oil}}/\omega^I_{\text{oil}} = \ell_E^2/\ell_I^2 > 1$. Both components of the welfare differential are positive, and both increase with oil dependence $\phi_E \eta$.

Optimal reserves. The CDB provides a structural formula for the exporter's disaster-driven currency risk. An oil-exporting central bank should hold reserves proportional to

$$R^* \propto \text{CDB} \times \lambda(\bar{G}) \times \text{GDP}_E \tag{44}$$

An exporter facing a higher CDB or higher baseline disaster intensity should hold proportionally more reserves. The model does not endogenize the reserves decision, but the CDB provides the risk quantity that a reserves-optimization model would take as input.

Hedging. The supply diffusion component of the risk premium can be hedged with oil futures; the disaster component requires tail-risk instruments. The welfare cost of geopolitical escalation is $d\omega^E/dG = \Phi^E \lambda_1/(\delta\chi) \approx 2\%$ of consumption per unit increase in G_t.

5.5 Comparative statics

EZ preferences create a clean separation: the IES ψ affects only the carry channel; supply volatility σ_S affects only the between-disaster supply channel; risk aversion γ affects all channels. Under CRRA ($\gamma = 1/\psi$), the carry channel vanishes entirely. Oil dependence η scales the differential $\ell_E - \ell_I = \phi_E \eta$ linearly, strengthening all results monotonically; at $\eta = 0$ the exchange rate is constant. The demand channel Δ_D decreases in the elasticity of substitution σ, and the sign reversal threshold G^{**} exists only for $\sigma < \sigma^*$.

5.6 Limitations

Complete markets and endowment economy. The base model has Arrow-Debreu markets. Section 4 relaxes the asset span to bonds and oil futures, showing that futures span both diffusion and jump risk (Lemma 11) and that the spanning wedge is quantitatively negligible for the carry premium. The welfare-premium disconnect (11.6% of permanent consumption versus less than 1 bp/yr) implies that the carry premium poorly measures welfare consequences of market incompleteness. The model has no production, investment, or labor market; the endowment specification isolates preference and endowment channels but cannot speak to employment or GDP effects.

Calibration tensions. No single γ simultaneously matches FX volatility and the carry premium, a tension inherited from Farhi and Gabaix (2016) and Wachter (2013). The CES specification amplifies all moments through $\varepsilon_p > 1$: at $\sigma = 0.25$, the oil-currency correlation sign is correct and oil price volatility matches, but FX volatility overshoots. Matching all moments simultaneously requires additional structure beyond the endowment economy. Disasters in (1) are permanent, but Remark 2 (Appendix) shows that temporary disasters preserve the sign structure with attenuation factor $\mu_R/(\mu_R + \delta) \approx 0.99$ for one-year average recovery.

Oil dependence parameter. Regressing non-oil GDP growth on oil GDP growth (constructed from World Bank oil rents data, 2000–2022) gives country-specific η estimates: Russia 0.32 ($p < 0.001$), Norway 0.26 ($p = 0.008$), Brazil 0.26 ($p = 0.002$), Colombia 0.20 ($p = 0.011$), Mexico 0.15 ($p = 0.046$), Canada 0.04 ($p = 0.53$). Saudi Arabia has near-zero η (0.02, $p = 0.45$) despite high oil share, likely reflecting sovereign wealth fund buffering. The cross-country ordering broadly matches the calibration in Table 4. The power-law specification $Y_t^{N,E} \propto (Y_t^O)^\eta$ does not emerge from a production function; alternative functional forms preserve the sign structure because the key requirement is $\ell_E > \ell_I$.

6 Conclusion

A two-country model with CES preferences, Epstein-Zin recursive utility, oil supply jump-diffusion, and asymmetric disaster exposure generates a state-dependent oil-currency correlation. When oil and the numeraire are complements ($\sigma < \sigma^*$), demand shocks produce a positive oil-currency covariance through expenditure-share amplification, and the total correlation shifts from positive in calm times to negative under geopolitical stress as the disaster channel dominates. The Cobb-Douglas special case ($\sigma = 1$) produces only a negative supply-driven covariance and misses the sign reversal. The commodity disaster beta aggregates oil dependence, risk aversion, and disaster severity into a quantity that replaces the free Farhi and Gabaix (2016) resilience parameter and prices the exporter's currency risk. Under incomplete markets, oil futures span both diffusion and jump risk from the oil price, but the residual spanning wedge is too small to resolve the dual-γ tension. Sanctions that remove futures access create welfare costs exceeding 11% of permanent consumption while shifting the carry premium by less than 1 basis point per year, revealing a sharp disconnect between the welfare consequences of market incompleteness and its asset-pricing footprint.

A Appendix

A.1 Equilibrium existence and integrability

The equilibrium existence results of Duffie and Skiadas (1994) and Schroder and Skiadas (1999) require that endowment processes have finite moments under the physical measure. The key condition is that the expected discounted value of the endowment stream is finite:

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}(Y_t^i)^{1-1/\psi}\,dt\right] < \infty$$

for each country i.

Boundedness of the intensity. The intensity $\lambda(G_t) = \lambda_0 + \lambda_1 G_t$ is linear in the OU process G_t, which has stationary distribution $N(\bar{G}, \sigma_G^2/(2\kappa))$. At the baseline, the stationary standard deviation is $\sigma_G/\sqrt{2\kappa} = 0.55$, giving $\lambda(G_t) < 0.023$ with probability exceeding 0.999.

Integrability. The expected number of disasters over $[0, T]$ is $(\lambda_0 + \lambda_1\bar{G})T < \infty$. The oil endowment Y_t^O is a geometric jump-diffusion with $\mathbb{E}[(1-Z)^p] < 1$ per disaster for $p = 1 - 1/\psi > 0$. Combined with the discount factor $e^{-\beta t}$, the integral converges for $\beta > g_O$, which holds at baseline. The CES utility satisfies the regularity conditions in Schroder and Skiadas (1999), and Karatzas and Shreve (1991) Theorem 11.2.10 ensures the SDF exists for bounded-in-probability intensity processes.

A.2 Consumption loadings derivation

Importer. With CES utility, $C^I = I^I \cdot \Pi(p)^{-1}$ where $\Pi(p) = [\alpha p^{1-\sigma} + (1-\alpha)]^{1/(1-\sigma)}$. Shephard's lemma gives $d\ln\Pi/d\ln p = s_O$, so $c_I^S = s_O\varepsilon_p$ and $c_I^D = 1 - s_O\varepsilon_p$. Under Cobb-Douglas: $c_I^S = \alpha$ and $c_I^D = 1-\alpha$. At disaster: $\Delta\ln C_t^I = s_O\varepsilon_p^{\text{jump}}\ln(1-Z) = \ell_I\ln(1-Z)$.

Exporter. Under the small exporter approximation, oil revenue $p_t Y_t^O \approx s_O Y_t^{N,I}$. The oil revenue elasticities are $(1-\varepsilon_p)$ to supply and $[(1-\sigma)(1-s_O)\varepsilon_p + 1]$ to demand. Income-share-weighted aggregation gives:

$$\Delta_S = \phi_E\eta - (1-\phi_E)(\varepsilon_p - 1), \qquad \Delta_D = (1-\phi_E)(1-\sigma)(1-s_O)\varepsilon_p - \phi_E$$

At disaster: $\Delta\ln C^E = (\phi_E\eta + \alpha)\ln(1-Z) = \ell_E\ln(1-Z)$.

A.3 Small exporter approximation: error analysis

The approximation replaces $Y_t^N = Y_t^{N,E} + Y_t^{N,I}$ with $Y_t^N \approx Y_t^{N,I}$, introducing a relative error $r \equiv Y_t^{N,E}/Y_t^{N,I}$. The correction to the oil-currency covariance is $\gamma\phi_E(1-\phi_E)\eta\Delta_S\sigma_S^2 = O(\phi_E^2)$, because the first-order effect feeds back through the same channel, creating a quadratic interaction. At $\phi_E = 0.6$ under Cobb-Douglas: the approximate covariance is -0.0054, the correction is $+0.00065$ (12% relative error), and the sign is preserved. Under CES ($\sigma = 0.25$), the sign is also preserved.

A.4 The f_V correction

Lemma 16 (f_V Correction)**.** *Under the log-linear approximation for ν_t^i, the effective carry sensitivity including the f_V correction is $\beta_e^{\text{eff}} = \beta_e(1 + \eta_{fV})$ where $\eta_{fV} > 0$.*

Proof. Let $h(G) \equiv (1-\gamma)V/C^{1-\gamma}$ and write $h^i(G) = h_0^i[1 + \Phi^i\lambda(G)/(\delta\chi)]$ to first order. The difference $f_V^I - f_V^E$ is proportional to $(\Phi^I - \Phi^E)\lambda(G)/(\delta\chi)$, the same factor driving the wealth-consumption ratio channel. Define

$$\eta_{fV} = \frac{\beta(1-\gamma)(1/\theta - 1)(h_0)^{-1/\theta}}{(1-1/\psi)\theta(1/\psi - \gamma)}$$

With $\gamma > 1/\psi$: both numerator and denominator are positive (since $1-\gamma < 0$, $1/\theta - 1 < 0$, $\theta < 0$, $1/\psi - \gamma < 0$), so $\eta_{fV} > 0$. At $\gamma = 5$, $\psi = 2$: $\eta_{fV} \approx 0.009$ (less than 1%). Under CES with $\sigma = 0.25$: $\eta_{fV} \approx 0.016$ (below 2%), because ε_p amplifies both numerator and denominator of η_{fV} proportionally. □

A.5 Temporary disasters

Remark 2 (Temporary Disasters)**.** *Suppose that after a disaster of size Z, oil supply recovers at rate $\mu_R > 0$: $Y_{t+s}^O = Y_{t^-}^O(1-Z)e^{\mu_R s}$ for $s > 0$. The disaster resilience becomes*

$$\Phi^i(\mu_R) = \frac{\mu_R}{\mu_R + \delta}\left(\mathbb{E}[(1-Z)^{-\gamma\ell_i}] - 1\right) + \frac{\mu_R}{\mu_R + \delta} - 1 \tag{45}$$

The factor $\mu_R/(\mu_R + \delta)$ discounts the permanent resilience by the expected duration of the disruption. The sign structure is preserved: $\Phi^E(\mu_R) > \Phi^I(\mu_R)$ for any $\mu_R > 0$ because $(1-Z)^{-\gamma\ell_E} > (1-Z)^{-\gamma\ell_I}$ for all $Z > 0$, and the $\mu_R/(\mu_R + \delta)$ factor is common. Therefore $\beta_e > 0$, the disaster crash $\Delta \ln e_t < 0$, and the CDB remains positive. The magnitudes attenuate by $\mu_R/(\mu_R+\delta)$, which equals 0.99 for one-year average recovery ($\mu_R = 1$, $\delta \approx 0.01$).

A.6 Special cases

The model nests several known frameworks:

Farhi-Gabaix (Case 3). Setting $\ell_I \to 0$, $\sigma_S \to 0$, and $\sigma_D \to 0$ eliminates oil market dynamics. The resilience $\Phi^I \to 0$, and the model reduces to Farhi and Gabaix (2016) with the exporter as the risky country: $\beta_e \to (\gamma - 1/\psi)\Phi^E \lambda_1/(\delta\chi)$.

CRRA (Case 1). Setting $\gamma = 1/\psi$ gives $\theta = 1$ and $\beta_e = 0$. The geopolitical carry channel vanishes. The exchange rate between disasters is driven only by oil supply and demand shocks. Disaster crashes still occur.

Symmetric countries (Case 4). Setting $\eta = 0$ gives $\ell_E = \ell_I = \alpha$, $\Phi^E = \Phi^I$, and $\beta_e = 0$. The exchange rate is constant. Oil shocks are common factors.

No disaster risk (Case 5). Setting $\lambda_0 = \lambda_1 = 0$ eliminates disasters. The exchange rate is driven only by oil supply and demand diffusion. Under CES with $\sigma < \sigma^*$, the demand channel generates a positive oil-currency correlation, but there is no sign reversal because the negative disaster contribution is absent.

Cobb-Douglas limit of CES (Case 6). Setting $\sigma = 1$ and $\sigma_D = 0$ in the CES model recovers the Cobb-Douglas baseline. The demand channel contribution is $\Delta_D = -\phi_E < 0$: the knife-edge case where the expenditure share is constant.

A.7 CES consumption loadings

The importer's indirect utility is $C^I = I^I \cdot \Pi(p)^{-1}$ with CES price index $\Pi(p) = [\alpha p^{1-\sigma} + (1-\alpha)]^{1/(1-\sigma)}$. Shephard's lemma gives $d\ln\Pi/d\ln p = s_O$, so $c_I^D = 1 - s_O\varepsilon_p$ and $c_I^S = s_O\varepsilon_p$. The exporter's oil revenue elasticities are $(1-\varepsilon_p)$ to supply and $[(1-\sigma)(1-s_O)\varepsilon_p + 1]$ to demand. Income-share-weighted aggregation yields the differentials Δ_S and Δ_D stated in (22)–(23).

A.8 Welfare cost of geopolitical risk

The welfare expressions $\omega^i(G) \approx \Phi^i\lambda(G)/(\delta\chi)$ and $\omega^i_{\text{oil}} \approx \gamma\ell_i^2\sigma_S^2/(2\delta)$ follow from the certainty-equivalent calculation under EZ preferences with the log-linear wealth-consumption ratio approximation. Both components satisfy $\omega^E > \omega^I$ and increase in oil dependence $\phi_E\eta$.

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